Hard Metrics from Cayley Graphs

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Metric distortion and hard metrics

The distortion of embedding a metric μ into an Euclidean space, $c_2(\mu)$, is defined as the minimum contraction over all non-expanding embeddings of μ into L_2 .

A fundamental result of Bourgain[85] claims that for any metric μ on n points,

$$c_2(\mu)=O(\log n).$$

A complementary result of LLR[95] and AR[98] claims that there exists metrics μ such that $c_2(\mu)=\Omega(\log n)$.

We call such metrics hard.

Examples of hard metrics

In what follows, we restrict ourselves to shortest-path metrics μ_G of undirected graphs G.

It was shown in LLR[95] and AR[98] that when G is a constant degree expander, μ_G is hard.

This remained (essentially) the only known construction of hard metrics until the work of KN[04].

Many fundamental results in the Theory of Finite Metric Spaces claim that certain important classes of metrics (e.g., planar, doubling, NEG) are not hard.

Results of this paper

We present other construction of hard metrics.

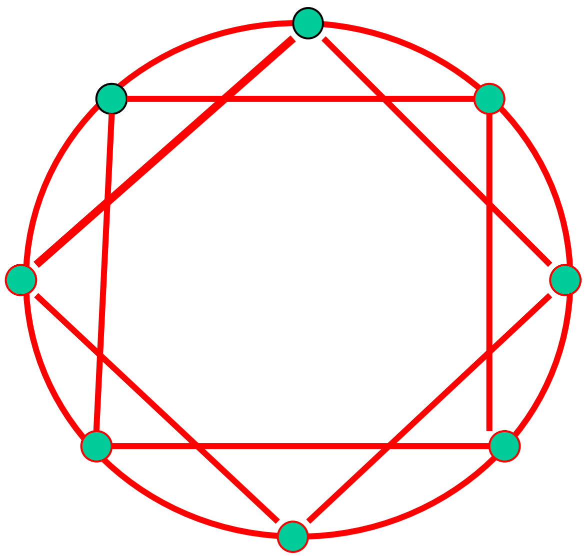
It is substantially different from LLR[95], AR[98].

Despite apparent lack of similarity with KN[05], our construction turns out to be more general, while being conceptually simpler.

The construction is based on Cayley Graphs of Abelian groups.

Cayley Graphs: A reminder

Let H be an Abelian group, and $A=-A$ a set of generators of H. The Cayley graph $G(H,A)$ has $V=H$, and $E=\{(x,y) \mid x-y \text{ is in } A\}$.



The Cayley Graph of Z_8 with generators $A=\{1,2,6,7\}$.

Lower bounds on distortion

Let $G=(V,E)$ be a graph of degree d, and let μ_G be its shortest-path metrics. We want to lower-bound $c_2(\mu_G)$.

A Poincare form: $F(\delta) = \sum_E \delta^2(i,j) \,/\, \sum_{V\times V} \delta^2(i,j)$.

Let $X= F(\mu_G)$, and $Y= \min\{F(\delta) \mid \delta \text{ is Euclidean}\}$. Then, $\{c_2(\mu_G)\}^2 \geq Y/X$.

In our case, $X \approx E/(n^2\, \mathrm{Diam}^2(G))$ and $Y = \gamma_G/n$, where γ_G is the spectral gap of G. Thus,

$$\{c_2(\mu_G)\}^2 \geq \gamma_G/d\, \mathrm{Diam}^2(G).$$

Hard graphs

Thus, in order to get $c_2(\mu_G)=\Omega(\log n)$, it suffices to require constant relative spectral gap γ_G/d, and **Diam(G)=Ω(log n)**.

Clearly, const. degree expanders achieve this. Are there other "hard" graphs?

Consider Cayley graphs of Abelian groups. It is well known that in this case γ_G/d cannot be constant unless for $d=\Omega(\log n)$.

This appears to be a problem: typically, a non-constant degree yields sub-logarithmic Diam…

Hard Metrics form Cayley graphs of Abelian groups

However, in bounding the Diam, the commutativity is our ally!

In particular, the number of vertices reachable from a fixed vertex in r steps is at most (r+d-1 choose r), the number of multisets of size r formed by d distinct elements.

Consequently, for any such graph G with n vertices and O(log n) degree, Diam(G)=Ω(log n).

It remains to take care of the normalized spectral gap γ_G/d . We need it to be constant.

Cayley graphs of Abelian groups (cont.)

A well known result of AR[94] claims that for any group H and a random set of generators A of size ≥ c log n, the corresponding Cayley graph G(H,A) almost surely has constant γ_G/d.

(For an Abelian H this is a mere exersize...)

Combining our observations, we arrive at:

Theorem For any Abelian group H, and a random (symmetric) set of generators A of size c log n, the shortest-path metric of the corresponding Cayley graph G(H,A) is almost surely hard.

When $H=(Z_2)^n$

In this case, the graph $G(H,A)$ has constant γ_G/d iff the matrix M_{nxm} whose set of columns is A is a generator matrix of linear error-correcting code with linear distance.

Since there exists such codes of constant rate, i.e., $m=O(n)$, we conclude that

Theorem* Let M be a generator matrix of a linear code of constant rate and linear distance, and let A be the set of M's columns. Then the shortest-path metric of the Cayley graph $G(H,A)$ is hard.

Conclusion

Hard metrics is a very interesting class of metric spaces with extremal properties. It is closely related to expanders and optimal error-correcting codes.

While the present work contributes to a better understanding of hard metrics, much remains to be done.

It is our hope that gradually the structure of hard metrics will become (reasonably) clear.