

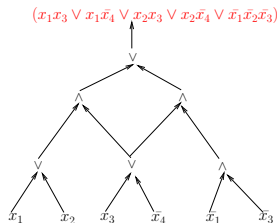
Arithmetizing Circuits around NC^1 and L

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Joint work with
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Boolean Circuits



Definition

Directed acyclic graph where nodes labeled with $\{\vee, \wedge, \neg, 0, 1, x_1, \dots, x_n\}$.

- A node of out-degree zero, called **output** node of the circuit
- $\{x_1, \dots, x_n\}$ are the inputs for the circuit, where $x_i \in \{0, 1\}$
- **fan in**(**fan out**) of a node is its in-degree(out-degree)
- **depth** - length of longest path from output node to input node
- **width** - maximum number of nodes at any particular level

Definition

- Class of problems which can be decided by $O(\log n)$ depth, poly size, constant fan-in circuits
- Examples:
 - ▶ parity of n bits,
 - ▶ sorting n numbers,
 - ▶ evaluating a boolean formula
- Upper bound: $NC^1 \subseteq DLOG$

Classes Equivalent to NC^1

- Bounded Width Branching Programs : **BWBP**
- Bounded Width Circuits : **BWC**
- Log Width Formula : **LWF**
- Branching Program over NFA : **BP-NFA**
- Branching Program over Visibly Pushdown Automata : **BP-VPA**

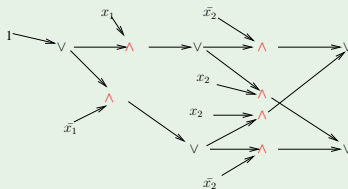
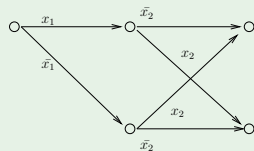
Bounded Width Circuits

Definition (SC^i)

$SC^i = DTIME-SPACE(poly, \log^i n) = CircuitSize, Width(poly, \log^i n)$

- $BWC = SC^0$ (by definition)
- $BWC \subseteq NC^1$ (divide-and-conquer)
- $BP = skew\text{-circuits}$, hence $BWBP \subseteq BWC$

Example



Log Width Formula

Definition (Formula)

A circuit where fan out is at most one (i.e a tree)

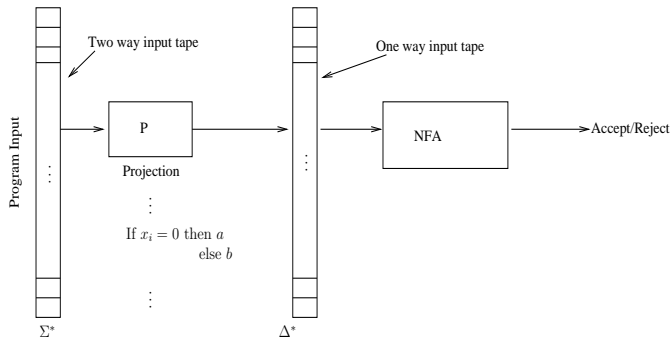
Definition (LWF)

Logarithmic Width Formula.

Theorem (Istrail, Zivkovic '94)

$$NC^1 = LWF$$

BP-NFA



Theorem (Barrington '89)

$$BWBP = BP-NFA$$

Visibly Pushdown Automaton: VPA

Definition (Mehlhorn '80... Alur, Madhusudan'04)

A Visibly Pushdown Automaton is a

- Pushdown automaton M , with input alphabet Δ partitioned as $(\Delta_{push}, \Delta_{pop}, \Delta_{int})$
- M 's action is guided by the input alphabet

Example

- Language $\{a^n b^n \mid n \geq 0\}$ can be recognized by VPAs.

Theorem (follows from Dymond '88)

$$BP\text{-VPA} \subseteq NC^1$$

Classes Equivalent to NC^1

NC^1	log depth, poly size circuits with fanin-2
$BWBP$	bounded width branching programs
BWC	bounded width circuits
LWF	log width formula
$BP-NFA$	programs over NFA
$BP-VPA$	programs over VPA

Are their arithmetic versions equivalent?

Arithmetization over \mathbb{N}

- Circuits

- ▶ Replace \wedge with \times and \vee with $+$.
- ▶ Counting number of proving subtrees

- Branching Programs

- ▶ Counting the number of s - t paths in a branching program
- ▶ Counting the number of accepting paths in M , for $BP\text{-}M(M = VPA, NFA)$

- Arithmetic classes:

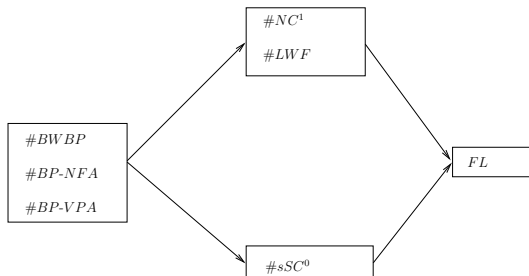
$\#NC^1$, $\#BWBP$, $\#LWF$, $\#BP\text{-}NFA$, $\#BP\text{-}VPA$, $\#BWC$

Theorem (Caussinus et al '98 ,Istrail Zivkovic '94)

$$\#BWBP = \#BP\text{-}NFA \subseteq \#NC^1 = \#LWF$$

Our Results

- $\#BP-VPA = \#BP-NFA$
- $\#BWC = \#SC^0$ needs restrictions to become interesting
- We propose a restriction $\#sSC^0$



Infeasibility of $\#BWC$

- A width two circuit can compute super exponential values



- poly degree \Rightarrow feasible values
- poly degree, poly size = $SAC^1 = LogCFL$

$$NLOG \subseteq LogCFL \subseteq NC^2$$

Degree of a circuit

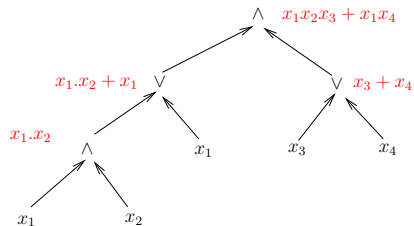


Figure: A Degree-3 circuit

Definition

$sSC^i =$ Poly degree, poly sized circuits of width $O(\log^i n)$

Observation: $NC^1 = sSC^0 = SC^0$

Open Question: $sSC^1 \stackrel{?}{=} SC^1 (= DLOG)$

Understanding sSC^1

- $NC^1 \subseteq sSC^1 \subseteq DLOG$
- Examine closure properties.

Is sSC^1 closed under complementation ?

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- $NC^1 \subseteq sSC^1 \subseteq DLOG$
- Examine closure properties.

Is sSC^1 closed under complementation ?

– Naive negation blows up the degree.

Theorem

$$\text{co-}sSC^i \subseteq sSC^{2i}$$

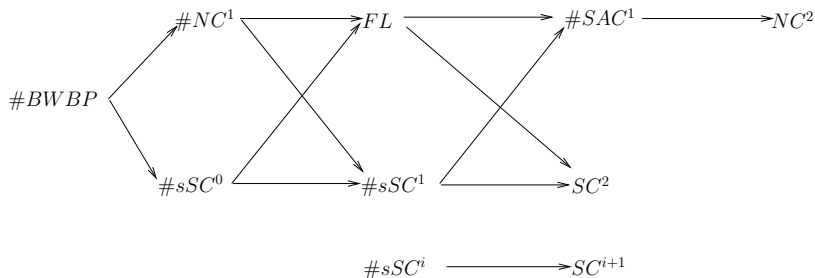
Proof.

Main ideas

- Inductive counting on layered circuits, as used by **Borodin, Cook, Dymond, Ruzzo, Tompa '89** to complement SAC^1 .
- Monotone branching programs for threshold, as used by **Vinay '96** to complement log width BPs .



Arithmetizing sSC - Overall picture



Open questions

- $sSC^1 \stackrel{?}{=} DLOG$
- $co-sSC^i \stackrel{?}{=} sSC^i, i > 0$
- $\#sSC^1 \stackrel{?}{\subseteq} FL$
- What closure properties do $\#sSC^i$ have ?
We show that $\#sSC^0$ is closed under div by a constant, decrement.