Computing representations of matroids of bounded branch-width

Daniel Kráľ
Institute for Theoretical Computer Science (ITI)
Charles University, Prague

MATROIDS, REPRESENTATION OF MATROIDS

- a matroid is a pair $(A, \mathcal{A})$, $\mathcal{A} \subseteq 2^A$

  A—the set of elements of the matroid

  $\mathcal{A}$—a hereditary set of independent sets

  Augmentation axiom: For every $X, Y \in \mathcal{A}$, $|X| < |Y|$,
  there exists $y \in Y$ such that $X \cup \{y\} \in \mathcal{A}$.

- edges of a graph, independent sets formed by acyclic subgraphs
- vectors of a vector space, independent sets formed by linearly independent vectors

- a matroid is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^d$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid
- graphic matroids are representable over all fields $\mathbb{F}$

Branch-width, analogue of tree-width

- graph tree-width is an important structural parameter
- a lot of intractable problems become tractable when restricted to graphs of bounded tree-width
- any graph MSOL property can be tested in polynomial time [Courcelle, 1990]

- analogous notion for matroids is branch-width
- elements of the matroid one-to-one correspond to the leaves of a tree with all inner nodes of degree three (branch-decomposition)
- the width of a decomposition is the maximum over all edges e of the tree of $\mathrm{rank}\, X + \mathrm{rank}\, Y - \mathrm{rank}\, M$ where X and Y are the elements separated by e
- the branch-width of a matroid is the minimum width of its decomposition

TRACTABILITY RESULTS

- the branch-width of a general matroid can be approximated in time $O(n^4)$ [Oum, Seymour 2006]

- matroids that can be represented over a finite field are close to graphs
- the branch-width of such a matroid can be computed in polynomial time [Hliněný 2005, Oum, Seymour 2006]
- MSOL properties of such matroids of bounded branch-width can be tested in polynomial time [Hliněný 2003]

Intractability results

- no subexponential algorithm can test whether an oracle-given matroid can be represented over GF(2) [Seymour 1981]
  the same remains true even if the given matroid has branch-width three

- it is NP-hard to decide whether a matroid of branch-width three represented over $\mathbb{Q}$ is representable over GF(p), $p \neq 2, 3$ [Hliněný 2006]

- the only good news. . .
  It can be decided in polynomial time whether a matroid represented over $\mathbb{Q}$ is representable GF(2) [Seymour 1980]

Our contribution

- How far the tractability results on computing the representations of matroids of bounded branch-width can be pushed?

- It can be tested in polynomial time whether a given matroid of bounded branch-width represented over a fixed finite field can be represented over another fixed finite field.

  If so, a representation can be found in polynomial time.
  All the representations can be listed in time linear in their number.

- A polynomial time algorithm for deciding whether a represented matroid of bounded branch-width can be represented over another finite field follows from previous results (MSOL property, Rota's conjecture).

Algorithm - the first step

- let $X \cup Y$ be a partition of a matroid $(A, \mathcal{A})$
  two sets $X_1, X_2 \subseteq X$ are Y-indistinguishable if for every $Y' \subseteq Y$:

$$\text{rank } X_1 \cup Y' - \text{rank } X_1 = \text{rank } X_2 \cup Y' - \text{rank } X_2$$

- if X and Y form a k-separation, i.e., rank X + rank Y − rank $A \leq k$,
  then there are at most $|\mathbb{F}|^{k^2}$ classes of Y-indistinguishable subsets of X
- given a branch-decomposition, root it at any vertex and for every inner node u with two children u_1 and u_2, consider the partitions $X_1 \cup Y_1$ and $X_2 \cup Y_2$ defined by edges to the children
  construct an auxiliary matrix labeled with rows labelled by classes of Y_1-indistinguishable and columns by classes of Y_2-indistinguishable subsets
  the entry in the row corresponding to X'_1 and column corresponding to X'_2 is rank X'_1 + rank X'_2 − rank $X'_1 \cup X'_2$

Algorithm - the second step

- computing representability and representations
- if $X \cup Y$ is a partition of a matroid $(A, \mathcal{A})$ and $\varphi : A \to \mathbb{F}^d$ its representation over $\mathbb{F}$, then for any two Y-indistinguishable subsets X_1 and X_2 and any $Y' \subseteq Y$,

$$\dim \overline{(\varphi(X_1) \cap Z) \cup (\varphi(Y') \cap Z)} = \dim \overline{(\varphi(X_2) \cap Z) \cup (\varphi(Y') \cap Z)}$$

  where $Z = \varphi(X) \cap \varphi(Y)$

- a type of a representation of X is the tuple of subspaces of Z generated by classes of Y-indistinguishable subsets of X
- since there are finitely many subspaces of Z and finitely many classes of Y-indistinguishable subsets of X, we can recursively compute the number of representations of each given type
- an actual representation can be found in an analogous way

Open problems

- Is it possible to test in polynomial time whether a matroid represented over $\mathbb{Q}$ is representable over GF(3)?
  At least, if the input matroid has branch-width at most $k \geq 3$.

- Can a representation over a fixed finite field $\mathbb{F}$ be found in polynomial time if a matroid is given by its representation over $\mathbb{Q}$, is guaranteed to be representable over $\mathbb{F}$, and has a bounded branch-width?

- Can a bijection φ of elements of two matroids of bounded branch-width be tested to be an isomorphism in polynomial time if both the matroids are represented over $\mathbb{Q}$ and have bounded branch-width?