Symmetries and the Complexity of Pure Nash Equilibrium

by

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Outline

Introduction

Motivation.

- Game theory
- Solution concepts (Nash equilibrium, iterated dominance, ...)

Complexity Results for Nash Equilibrium.

- PPAD complete (Chen and Deng, 2006; Daskalakis and Papadimitriou 2006)
- *Succinct representations*
  - Congestion games: PLS-complete (Fabrikant et al., 2004)
  - Graphical normal form: NP-complete (Gottlob et al., 2005; Fischer et al., 2006)
  - Circuit form: NP-complete (Schoenebeck and Vadhan, 2006)

Here.

- Complexity of restricted normal-form games

Normal-Form Games

Normal-Form Game. $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ with

- N a set of *players*
- A_i a nonempty set of *actions* for player i
- $p_i : (\times_{j \in N} A_j) \to \mathbb{R}$ a *payoff function* for player i

Example. Prisoners' Dilemma, Matching Pennies

	C	D
C	$(2,2)$	$(0,3)$
D	$(3,0)$	$(1,1)$

	H	T
H	$(1,0)$	$(0,1)$
T	$(0,1)$	$(1,0)$

Strategy.

- (Pure) strategy $s_i \in \Delta(A_i)$: (degenerate) probability distribution over A_i
- *Strategy profile* $s \in \times_{i \in N} \Delta(A_i)$

Nash Equilibrium

Idea. Strategies that are *mutual best responses* to each other

Nash Equilibrium. Profile s is a *Nash equilibrium* if for every player $i \in N$, s_i is a *best response* to s_{-i}, i.e.,

$$p_i(s) \geq p_i((s_{-i}, a)),$$

for every $a \in A_i$, where $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ and $(s_{-i}, a) = (s_1, \dots, s_{i-1}, a, s_{i+1}, \dots, s_n)$

Matching Pennies.

	H	T
H	$(1,0)$	$(0,1)$
T	$(0,1)$	$(1,0)$

Equilibrium. Mixing with prob. $\frac{1}{2}$; no pure equilibrium.

Remark. General existence theorem (Nash, 1951); *pure* strategy profile *not* guaranteed to exist

Symmetries in Normal-Form Games

Idea. Exploit similarities between players

Forms of Symmetry.

- Players cannot or need not distinguish between other players
- Players do not distinguish *themselves* from the other players

This leads to

symmetry and anonymity

combined with (strong) or without (weak) identical payoff functions for all players

Remark. Representation has polynomial size *in general* if and only if number of actions k is a constant

Symmetric Normal-Form Game

A game $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ with $A_i = A$ for all $i \in N$ is

- *weakly symmetric*, if

$$p_i(s) = p_i(t) \qquad \text{for all } i \in N \text{ and all } s, t \in A^N \text{ with } s_i = t_i \text{ and } \#(s_{-i}) = \#(t_{-i}),$$

  where $\#(s) = (\#(a, s))_{a \in A}$ is the commutative image (or Parikh image) of action profile s, and

- *strongly symmetric*, if

$$p_i(s) = p_j(t) \qquad \text{for all } i, j \in N \text{ and all } s, t \in A^N \text{ with } s_i = t_j \text{ and } \#(s_{-i}) = \#(t_{-j}).$$

Anonymity. (i) Skip $s_i = t_i$ and $s_i = t_j$ and (ii) replace s_{-i} by s and t_{-i} by t (same with t_{-j}).

Examples

Players 1, 2, and 3 choose rows, columns, and tables, respectively.

Weakly Symmetric.

$(\cdot,\cdot,\cdot)$	(a,g,c)		$(a,b,\cdot)$	$(\cdot,e,f)$
$(\cdot,b,c)$	$(d,e,\cdot)$		$(d,\cdot,f)$	$(\cdot,\cdot,\cdot)$

Strongly Symmetric.

(a,a,a)	(b,c,b)		(b,b,c)	(e,d,d)
(c,b,b)	(d,d,e)		(d,e,d)	(f,f,f)

Weakly Anonymous.

$(\cdot,\cdot,\cdot)$	(a,b,c)		(a,b,c)	(d,e,f)
(a,b,c)	(d,e,f)		(d,e,f)	$(\cdot,\cdot,\cdot)$

Strongly Anonymous.

(a,a,a)	(b,b,b)		(b,b,b)	(c,c,c)
(b,b,b)	(c,c,c)		(c,c,c)	(d,d,d)

Nash Equilibria in Symmetric Games

Known Results.

- Every (strongly) symmetric game has a *symmetric* equilibrium (Nash, 1951)
- Symmetric equilibrium can be computed in P if $|A| = O(\log |N| / \log \log |N|)$ (Papadimitriou and Roughgarden, 2005)

Problems.

- Does not apply to pure equilibria or weak symmetry
- Not obvious that symmetry simplifies the search for equilibria

(0, 1, 1)	(0,0,1)
(1,1,1)	(0, 0, 0)

(0,1,0)	(0, 0, 0)
(0, 1, 0)	(1, 0, 1)

Complexity Results

Theorem (Complexity of Restricted Normal-Form Games)

	$|A| = O(1)$	$|A| = O(|N|)$
weakly symmetric	TC^0-complete	*NP-complete*
weakly anonymous	TC^0-complete	*NP-complete*
strongly symmetric	*in* AC^0	*NP-complete*
strongly anonymous	*in* AC^0	*PLS-complete*

Complexity Classes.

- AC^0: Boolean circuits with constant depth, unbounded fan-in, polynomial size
- TC^0: AC^0 plus threshold gates
- $AC^0 \subset TC^0 \subseteq P \subseteq NP$
- PLS: polynomial local search

Skip proof for $|A| = O(1)$ ...

Weak Symmetry/Anonymity, $|A| = O(1)$

Theorem

Deciding whether a weakly symmetric or weakly anonymous game with a constant number of actions has a pure Nash equilibrium is TC^0-complete under constant-depth reducibility. Hardness holds even if there is only a constant number of payoffs and only two different payoff functions.

Skip proof ...

Sketch of Proof. Membership in TC^0. Fix a particular $x = \#(s)$ with $s \in A^N$. Note that constant $|A|$ implies

- constant number of subsets $C \subseteq A$, and
- x takes only polynomially many different values.

Hence certainly in P; membership in TC^0 can be shown by exploiting the algorithm on the next slide in more detail.

Then do the following:

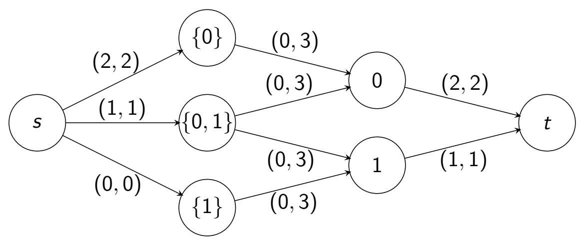
1. For each $C \subseteq A$, compute the number w_C of players for which C is the set of *potential* best responses under x

(0, 1, 1)	(0,0,1)
(1,1,1)	(0, 0, 0)

(0,1,0)	(0, 0, 0)
(0, 1, 0)	(1, 0, 1)

   Here $w_{\{0\}} = 2$, $w_{\{0,1\}} = 1$, and $w_{\{1\}} = 0$ for $x = (2, 1)$.

2. Check whether the numbers $(w_C)_{C \subseteq A}$ are "compatible" with x (reduces to a flow problem of constant size)



   Solution $w_{(\{0\},0)} = 2$, $w_{(\{0,1\},0)} = 0 = w_{(\{1\},1)}$, and $w_{(\{0,1\},1)} = 1$.

TC^0-hardness by a reduction from MAJORITY. Design a game Γ with $m+2$ players of two types 0 and 1 and actions $A = \{0, 1\}$. Define the payoffs as follows:

	0		...			$\ell+1$			...		$m+2$
p_0	...	0	1	0	2	1	0	1	0	1	...
p_1	...	1	0	1	0	1	2	0	1	0	...

where type of player i depends on value of ith input bit, players $m+1$ and $m+2$ are of of type 0 and 1, respectively.
Observe:

- A profile s cannot be a Nash equilibrium if $\#(1, s) \neq \ell + 1$.
- If there are $\ell + 1$ players of type 1, the action profile where all players of type 0 play action 0 and all players of type 1 play action 1 is a Nash equilibrium.
- If the number of players of type 1 does not equal $\ell + 1$, a profile s with $\#(s, 1) = \ell + 1$ cannot be an rquilibrium. □

Strong Symmetry/Anonymity, $|A| = O(1)$

Theorem

The problem of deciding whether a strongly symmetric game with a constant number of actions has a pure Nash equilibrium is in AC^0.

Theorem

The problem of finding a social-welfare-maximizing pure Nash equilibrium of a strongly anonymous game with a constant number of actions is in AC^0.

Remark. Unlike *weak* symmetry/anonymity: If s is a Nash equilibrium, so are all t with $\#(t) = \#(s)$. Hence we only need to check best response property for player *playing a certain action*, of which there are at most $|A|$.

Strong Symmetry/Weak Anonymity, $|A| = O(|N|)$

Theorem

Deciding whether a weakly anonymous or strongly symmetric game has a pure Nash equilibrium is NP-complete, even if the number of actions is linear in the numb er of players and there is only a constant number of different payoffs.

Skip proof ...

Sketch of Proof. Containment within NP by guess and check. Hardness by a reduction from Boolean circuit satisfiability problem. For a circuit $\mathcal{C}$ with inputs M design game Γ with players $N = M$ and actions $A = \{ a_i^0, a_i^1 \mid i \in M \}$. Action profile s satisfying

$$\#(a_i^0, s) + \#(a_i^1, s) = 1,$$

corresponds to an assignment of $\mathcal{C}$.

Define payoff p_i as follows:

- If s corresponds to a *satisfying* assignment of $\mathcal{C}$, then $p_i(s) = 2$ for all $i \in N$.
- If s corresponds to an assignment that does not satisfy $\mathcal{C}$, then
  - $p_1(s) = 2$, $p_2(s) = 1$ if $|\{ i \in M \mid \#(a_i^0, s) > 0 \}|$ is even, and
  - $p_1(s) = 1$, $p_2(s) = 2$ if this number is odd.
  - For all $i \in N \setminus \{1, 2\}$, we let $p_i(s) = 2$.
- If s does *not* correspond to an assignment of $\mathcal{C}$, then $p_i(s) = 1$, if $\#(a_i^0, s) + \#(a_i^1, s) > 0$, and $p_i(s) = 0$ otherwise.

Observe:

- Clearly, every action profile s corresponding to a satisfying assignment of $\mathcal{C}$ is a Nash equilibrium, because in this case all players receive the maximum payoff of 2.
- In any other case, s cannot be a Nash equilibrium. □

Strong Anonymity, $|A| = O(|N|)$

Theorem

The problem of finding a pure Nash equilibrium in a strongly anonymous game is PLS-complete, even if the number of actions is linear in the number of players.

Skip proof ...

Remark. Strongly anonymous games always have a pure Nash equilibrium

Sketch of Proof. Containment immedidate. Hardness by a reduction from the PLS-complete problem FLIP:

Input: Boolean circuit C with inputs $M = \{1, 2, \ldots, m\}$.

Search: Assignment such that the output interpreted as an ℓ-bit binary number is a local maximum under the FLIP neighborhood.

Design a normal-form game Γ with players $N = M$ and actions $A = \{ a_i^0, a_i^1 \mid i \in M \}$. Recall that action profile s with $\#(a_i^0, s) + \#(a_i^1, s) = 1$, corresponds to an assignment of $\mathcal{C}$.

Define payoff function p as follows:

- If s corresponds to an assignment c of $\mathcal{C}$, then $p(s) = |N| + \mathcal{C}(c)$.
- Otherwise, let $p(s) = |\{ i \in M \mid \#(a_i^0, s) + \#(a_i^1, s) > 0 \}|$.

Observe:

- There is a direct correspondence between Nash equilibria of Γ and local maxima of $\mathcal{C}$ under the FLIP neighborhood (changing between a_i^0 and a_i^1).
- Action profile s that does not correspond to a valid assignment of $\mathcal{C}$ cannot be a Nash equilibrium. □

Discussion

Summary.

- Four notions of symmetry in multi-player games
- Finding pure Nash equilibria is tractable if the number of actions is a constant; growing number of actions makes it intractable
- Identical payoff functions for all players simplify the problem
- Anonymity seems to have no influence on the complexity

Future work.

- Games with a slowly growing number of actions
- Mixed equilibria in weakly symmetric/anonymous games
- ...